

SCALABILITY OF FRAMES GENERATED BY DYNAMICAL OPERATORS

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ABSTRACT. Let A be an operator on a separable Hilbert space \mathbb{H} , and let $G \subset \mathbb{H}$. It is known that - under appropriate conditions on A and G - the set of iterations $F_G(A) = \{A^j \mathbf{g} \mid \mathbf{g} \in G, 0 \leq j \leq L(\mathbf{g})\}$ is a frame for \mathbb{H} . We call $F_G(A)$ a dynamical frame for \mathbb{H} , and explore further its properties; in particular, we show that the canonical dual frame of $F_G(A)$ also has an iterative set structure.

We explore the relations between the operator A , the set G and the number of iterations L which ensure that the system $F_G(A)$ is a scalable frame. We give a general statement on frame scalability, and study in detail the case when A is a normal operator, utilizing the unitary diagonalization in finite dimensions. In addition, we answer the question of when $F_G(A)$ is a scalable frame in several special cases involving block-diagonal and companion operators.

1. INTRODUCTION

The problem of generating frames by iterative actions of operators [7, 11, 8] has emerged within the research related to the dynamical sampling problem [1]-[11]. The conditions under which a frame generated by iterative actions of operators exists for a finite-dimensional or a separable Hilbert space have been stated in [7] and [11]. If we have a frame, then a linear combination of a dual frame with the dynamically sampled coefficients reproduce the original signal. The natural follow-up questions to ask in this setup are: whether we can obtain a scalable frame under iterative actions, and if not, whether we can find a dual frame which preserves the dynamical structure.

Let A be an operator on a separable Hilbert space \mathbb{H} . We consider a countable set of vectors G in \mathbb{H} , and a function $L : G \rightarrow \mathbb{Z}_+$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Related to the iterated system of vectors

$$\{A^j \mathbf{g} \mid \mathbf{g} \in G, 0 \leq j \leq L(\mathbf{g})\}, \quad (1)$$

we answer the following two questions:

- (Q1) What conditions on A , G and L ensure that (1) is a scalable frame for \mathbb{H} ?

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(Q2) Assuming the system (1) is a frame for \mathbb{H} , can we obtain a dual frame for (1), perhaps by iterative actions of some operator?

The motivation for studying systems of type (1) comes from the *dynamical sampling problem* (DSP): Find sampling locations that allow the reconstruction of an unknown function \mathbf{f} from the scarce samples of \mathbf{f} , and its evolved states $A^n \mathbf{f}$. In the DSP, n represents time, and A^* is an evolution operator; for instance, A^* can represent the heat evolution operator, \mathbf{f} the temperature at time $n = 0$, and $(A^*)^n \mathbf{f}$ the temperature at time n . The DSP for the heat evolution operator was studied in [26, 27]; generalizations of the DSP and related applications can be found in [1]–[11].

More precisely, the DSP is as follows: Let the initial state of a dynamical system be represented by an unknown element $\mathbf{f} \in \mathbb{H}$. Say the initial state \mathbf{f} is evolving under the action of an operator A^* to the states $\mathbf{f}_j = A^* \mathbf{f}_{j-1}$, where $\mathbf{f}_0 = \mathbf{f}$ and $j \in \mathbb{Z}_+$. Given a set of vectors $G \subset \mathbb{H}$, one can find conditions on A , G and $L = L(\mathbf{g})$ which allow the recovery of the initial state \mathbf{f} from the set of samples $\{\langle A^{*j} \mathbf{f}, \mathbf{g} \rangle \mid \mathbf{g} \in G\}_{j=0}^{L(\mathbf{g})}$. In short, the problem of signal recovery via dynamical sampling is solvable if the set of vectors $F_A^L(G) := \{A^j \mathbf{g} \mid \mathbf{g} \in G\}_{j=0}^{L(\mathbf{g})}$ is a frame for \mathbb{H} , [7]. In frame theory it is known that every frame has at least one dual frame; if $F_A^L(G)$ is a frame for \mathbb{H} , and its dual frame elements are $\mathbf{h}_{\mathbf{g},j}$, then all $\mathbf{f} \in \mathbb{H}$ are reconstructed as

$$\mathbf{f} = \sum_{\mathbf{g} \in G} \sum_{j=0}^{L(\mathbf{g})} \langle \mathbf{f}, A^j \mathbf{g} \rangle \mathbf{h}_{\mathbf{g},j}. \quad (2)$$

If the frame $F_A^L(G)$ is *scalable*, then its dual frame elements are $w_{j,\mathbf{g}}^2 A^j \mathbf{g}$ for some *scaling coefficients* $w_{j,\mathbf{g}}$, and the reconstruction formula (2) is

$$\mathbf{f} = \sum_{\mathbf{g} \in G} \sum_{j=0}^{L(\mathbf{g})} w_{j,\mathbf{g}}^2 \langle \mathbf{f}, A^j \mathbf{g} \rangle A^j \mathbf{g}. \quad (3)$$

Notice that the frame coefficients in (2) are exactly the samples

$$\langle A^{*j} \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{f}, A^j \mathbf{g} \rangle. \quad (4)$$

Thus the set of samples $\{\langle A^{*j} \mathbf{f}, \mathbf{g} \rangle \mid \mathbf{g} \in G\}_{j=0}^{L(\mathbf{g})}$ is sufficient for the recovery of \mathbf{f} . Since (2) requires that the dual frame of $F_A^L(G)$ is known, unless the frame is scalable as in (3), it is significant to find the answers to questions (Q1) and (Q2).

1.1. Contribution and organization. In Section 2 we recall the notions of frames, scalable frames and, in particular, frames of iterative actions of operators, i.e., dynamical frames. In Section 3, we illustrate the dynamical nature of the canonical dual frame of (1) in Theorem 1, and the fusion frame structure of dynamical frames (Corollary 1). In Section 4 we give a characterization of scalability in Theorem 2, under the assumption that A is normal. Section 5 contains several generalized examples of frames and

scalable frames in lower dimensions, and we characterize frame scalability in \mathbb{R}^2 and \mathbb{R}^3 . In addition, we provide examples of operators which are not normal, yet generate scalable frames for \mathbb{R}^2 and \mathbb{R}^3 . Motivated by these results, we study block-diagonal operators, which combine low-dimensional frames into higher-dimensional frames (Theorem 6). In Section 6, we also provide examples of dynamical scalable frames, generated using companion operators [22] and generalized companion operators. In section 7 we give initial answers to question (Q3), addressing frame scalability when multiple operators are involved.

2. PRELIMINARIES

Frames are a generalization of orthonormal bases. For an orthonormal basis $\{\mathbf{f}_i\}_{i \in I}$ of \mathbb{H} , it holds

$$\mathbf{f} = \sum_{i \in I} \langle \mathbf{f}, \mathbf{f}_i \rangle \mathbf{f}_i \quad \text{for all } \mathbf{f} \in \mathbb{H}. \quad (5)$$

The uniqueness of representation (5) is not always an advantage. In applications such as image and signal processing, the loss of a single coefficient during data transmission will prevent the recovery of the original signal, unless we ensure redundancy via frame spanning.

Since finding a dual frame can be computationally challenging, one significant direction of current research has been on the construction of tight frames in finite dimensions [12, 28, 19, 14, 23]. A tight frame plays the role of its own dual, and provides a reconstruction formula as in (5) up to a constant. Recently, the theme of scalable frames has been developed as a method of constructing tight frames from general frames by manipulating the length of frame vectors. Scalable frames maintain erasure resilience and sparse expansion properties of frames [13, 16, 24, 25, 18].

First, let us review relevant definitions and known results. Throughout this paper \mathbb{H} denotes a separable Hilbert space. Given an index set I , a sequence $F = \{\mathbf{f}_i\}_{i \in I}$ of nonzero elements of \mathbb{H} is a *frame* for \mathbb{H} , if there exist $0 < A \leq B < \infty$ such that

$$A\|\mathbf{f}\|^2 \leq \sum_{i \in I} |\langle \mathbf{f}, \mathbf{f}_i \rangle|^2 \leq B\|\mathbf{f}\|^2 \quad \text{for all } \mathbf{f} \in \mathbb{H}. \quad (6)$$

In finite dimensions, we find it useful to express frames as matrices, so we abuse the notation of F as follows: when $\dim \mathbb{H} = n$, a frame $F = \{\mathbf{f}_i\}_{i \in I}$ for \mathbb{H} is often represented by a $n \times k$ matrix F , whose column vectors are \mathbf{f}_i , $i = 1, \dots, k$. The frame operator $S = FF^*$ is then positive, self-adjoint and invertible.

For each frame F there exists at least one *dual* frame $G = \{\mathbf{g}_i\}_{i \in I}$, satisfying

$$\mathbf{f} = \sum_{i \in I} \langle \mathbf{f}, \mathbf{f}_i \rangle \mathbf{g}_i = \sum_{i \in I} \langle \mathbf{f}, \mathbf{g}_i \rangle \mathbf{f}_i \quad \text{for all } \mathbf{f} \in \mathbb{H}. \quad (7)$$

The matrix equation $FG^* = GF^* = I$ is an equivalent expression to the frame representation (7). The set $\{\mathbf{g}_i = S^{-1}\mathbf{f}_i\}_{i \in I}$ is called the canonical dual frame.

Finding a dual frame can be computationally challenging; thus it is of interest to work with tight frames. We say that a frame is *A-tight* if $A = B$ in (6). In this case, the function reconstruction is simplified since the frame operator is the identity operator up to scalar multiplication. So, for an *A-tight* frame, we only need one frame for both analysis and reconstruction, as (7) becomes

$$\mathbf{f} = \frac{1}{A} \sum_{i \in I} \langle \mathbf{f}, \mathbf{f}_i \rangle \mathbf{f}_i = \frac{1}{A} FF^* \mathbf{f} \quad \text{for all } \mathbf{f} \in \mathbb{H}. \quad (8)$$

When $A = 1$, we call F a Parseval frame. If a frame $F = \{\mathbf{f}_i\}_{i \in I}$ is not tight, but we can find scaling coefficients $w_i \geq 0$, $i \in I$, such that the scaled frame $F_w = \{w_i \mathbf{f}_i\}_{i \in I}$ is tight, then we call the original frame F a *scalable* frame. We note that the notion of scalability of a frame is defined for a unit-norm frame in [16], but in this manuscript we do not require a scalable frame to be unit-norm. For a scalable frame, the scaled frame representation becomes

$$\mathbf{f} = \sum_{i \in I} \langle \mathbf{f}, w_i \mathbf{f}_i \rangle w_i \mathbf{f}_i = F_w F_w^* \mathbf{f} = F D_{w^2} F^* \mathbf{f} \quad \text{for all } \mathbf{f} \in \mathbb{H}, \quad (9)$$

where D_{w^2} denotes a diagonal operator with w_i^2 as diagonal entries. If the scaling coefficients w_i are positive for all $i \in I$, then we call the original frame F a *strictly scalable* frame.

Let I denote a finite or countable index set, let $G = \{\mathbf{f}_s\}_{s \in I} \subset \mathbb{H}$ and let $A : \mathbb{H} \rightarrow \mathbb{H}$ be a bounded operator. We call the collection

$$F_G^{\mathbf{L}}(A) = \cup_{s \in I} \{A^j \mathbf{f}_s : j = 0, 1, \dots, L_s\} \quad (10)$$

a *dynamical system*, where $L_s \geq 0$ (L_s may go to ∞) and $\mathbf{L} = (L_s)_{s \in I}$ is a sequence of iterations. The operator A , involved in generating the set (10), is sometimes referred to as a *dynamical operator*. If A is fixed, then we use the notation $F_G^{\mathbf{L}}$, and if $G = \{\mathbf{f}\}$ and $\mathbf{L} = \{L\}$, then we label (10) by $F_{\mathbf{f}}^L$.

Note that in [7], \mathbf{f}_s are chosen to be the standard basis vectors, while in this manuscript, we allow the use of any nonzero vector $\mathbf{f}_s \in \mathbb{H}$. If (10) is a frame for \mathbb{H} , then we call $F_G^{\mathbf{L}}(A)$ a *dynamical frame*, generated by operator A , set G and sequence of iterations \mathbf{L} .

3. NEW RESULTS ON DYNAMICAL FRAMES

As we are about to see in Theorem 1, the canonical dual frame of a dynamical frame preserves the dynamical structure, just like the canonical duals of wavelet or Gabor frames preserve the corresponding wavelet/Gabor structure [21].

Theorem 1. *Let $G = \{\mathbf{f}_s\}_{s \in I} \subset \mathbb{H}$, where I is a countable index set, and assume that $F_G^{\mathbf{L}}(A)$ is a frame for \mathbb{H} , with frame operator S . The canonical dual frame of $F_G^{\mathbf{L}}(A)$ is the dynamical frame $F_{G'}^{\mathbf{L}}(B)$, generated by $B =$*

$S^{-1}AS$, $G' = \{\mathbf{g}_s = S^{-1}\mathbf{f}_s\}_{s \in I}$, and sequence of iterations \mathbf{L} . That is, for every $\mathbf{f} \in \mathbb{H}$ the frame reconstruction formula is

$$\mathbf{f} = \sum_{s \in I} \sum_{j=0}^{L_s} \langle A^{*j}\mathbf{f}, \mathbf{f}_s \rangle B^j \mathbf{g}_s. \quad (11)$$

Proof. The elements of the canonical dual frame of $F_G^{\mathbf{L}}(A)$ are computed as $S^{-1}(A^j \mathbf{f}_s)$, $s \in I$, $j = 0, 1, \dots, L_s$. Let $\mathbf{g}_s = S^{-1}\mathbf{f}_s$, $s \in I$, then for all $j \geq 0$ we have

$$B^j \mathbf{g}_s = (S^{-1}AS)(S^{-1}AS) \dots (S^{-1}AS) \mathbf{g}_s = S^{-1}A^j(S\mathbf{g}_s) = S^{-1}(A^j \mathbf{f}_s),$$

and (11) follows by (7) and (4). \square

It is a known fact in frame theory that an invertible operator preserves the frame inequality. It follows from this that under the action of an invertible operator, the dynamical structure is preserved:

Theorem 2. Let \mathbb{H}_1 and \mathbb{H}_2 be two separable Hilbert spaces. Let $G = \{\mathbf{f}_s\}_{s \in I} \subset \mathbb{H}_1$, where I is a countable index set. Let $\mathbf{L} = (L_s)_{s \in I}$, $L_s \geq 0$. Let A be an operator on \mathbb{H}_1 and let $B : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be an invertible operator. Set $\mathbf{g}_s = B\mathbf{f}_s \in \mathbb{H}_2$, $s \in I$, and $C = BAB^{-1}$. TFAE:

- (i) The set $F = \cup_{s \in I} \{A^j \mathbf{f}_s\}_{j=0}^{L_s}$ is a frame for \mathbb{H}_1 ,
- (ii) The set $BF = \cup_{s \in I} \{C^j \mathbf{g}_s\}_{j=0}^{L_s}$ is a frame for \mathbb{H}_2 .

Proof. Let $\mathbf{g}_s = B\mathbf{f}_s \in \mathbb{H}_2$, $s \in I$, and set $C = BAB^{-1}$. Note that $C^j = BA^jB^{-1}$, due to $B^{-1}B = I$. For all $A^j \mathbf{f}_s \in F \subset \mathbb{H}_1$, we have

$$BA^j \mathbf{f}_s = BA^j B^{-1} B \mathbf{f}_s = BA^j B^{-1} \mathbf{g}_s = C^j \mathbf{g}_s \in BF \subset \mathbb{H}_2. \quad (12)$$

The operator B is invertible, thus BF is a frame if and only if F is a frame, so (i) and (ii) are equivalent. \square

Comment 1. If $\mathbb{H} = \mathbb{H}_1 = \mathbb{H}_2$, then Theorem 2 is a generalization of the change of basis result. Notice that under the action of an invertible operator $B : \mathbb{H} \rightarrow \mathbb{H}$, the elements of a dynamical frame F for \mathbb{H} preserve the dynamical structure, i.e., BF is also a dynamical frame for \mathbb{H} .

Fusion frames [15] are frames which decompose into a union of frames for subspaces of a Hilbert space \mathbb{H} . Given a countable index set I , let $\mathcal{W} := \{W_i \mid i \in I\}$ be a family of closed subspaces in \mathbb{H} . Let the orthogonal projections onto W_i be denoted by P_i . Then \mathcal{W} is a *fusion frame* for \mathbb{H} , if there exist $C, D > 0$ such that

$$C\|\mathbf{f}\|^2 \leq \sum_{i \in I} \|P_i(\mathbf{f})\|^2 \leq D\|\mathbf{f}\|^2 \quad \text{for all } \mathbf{f} \in \mathbb{H}.$$

Let $F_i = \{\mathbf{f}_{ij}\}_{j \in J_i}$ be a frame for W_i , $i \in I$, with frame bounds A_i, B_i . If $0 < A = \inf_{i \in I} A_i \leq \sup_{i \in I} B_i = B < \infty$, then [15]:

$\cup_{i \in I} F_i$ is a frame for \mathbb{H} if and only if $\{W_i\}_{i \in I}$ is a fusion frame for \mathbb{H} . (13)

If F_i denotes the frame matrix formed by the frame vectors for each W_i , and G_i contains the dual frame elements $\{\mathbf{g}_{ij}\}_{j \in J_i}$, then the fusion frame operator S is positive and invertible on \mathbb{H} , and for all $\mathbf{f} \in \mathbb{H}$, we have

$$\mathbf{f} = \sum_{i \in I} F_i G_i^* \mathbf{f} = \sum_{i \in I} G_i F_i^* \mathbf{f}. \quad (14)$$

By (13) and (14), a dynamical frame induces a fusion frame:

Corollary 1. *Let $F = \cup_{s \in I} \{A^j \mathbf{f}_s\}_{j=0}^{L_s}$ be a frame for \mathbb{H} . We introduce subspaces of \mathbb{H} by*

$$W_s = \overline{\text{span}\{A^j \mathbf{f}_s : 0 \leq j \leq L_s\}}, \quad \text{for all } s \in I. \quad (15)$$

Then $\{W_s\}_{s \in I}$ is a fusion frame of \mathbb{H} .

4. SCALABLE FRAMES GENERATED BY DYNAMICAL OPERATORS

Now, we study the scalability of frames of type (1). A prior result on this topic (see Theorem 8 in [8]) has restrictive requirements, and delivers a tight frame if the involved operator A is a contraction, i.e., $A^j \mathbf{f} \rightarrow 0$ for all elements \mathbf{f} in the studied Hilbert space. Our research results illuminate the fact that - in finite dimensions - obtaining a tight or a scalable frame is possible in many cases.

If the operator B occurring in Theorem 2 is unitary, then the property of scalability is preserved, and we have:

Corollary 2. *Let $G = \{\mathbf{f}_s\}_{s \in I} \subset \mathbb{H}$ and $\mathbf{L} = (L_s)_{s \in I}$, $L_s \geq 0$. Let A be a bounded operator on a separable Hilbert space \mathbb{H} . If B is a unitary operator on \mathbb{H} , then $\cup_{s \in I} \{A^j \mathbf{f}_s\}_{j=0}^{L_s}$ is a scalable frame if and only if $\cup_{s \in I} \{C^j \mathbf{g}_s\}_{j=0}^{L_s}$ is a scalable frame, where $C = BAB^*$ and $\mathbf{g}_s = B\mathbf{f}_s$, $s \in I$.*

Corollary 3. *Let A, R be two operators on a separable Hilbert space \mathbb{H} , and let U be a unitary operator on \mathbb{H} . Let $\mathbf{f}_s \in \mathbb{H}$, and set $\mathbf{v}_s = U^* \mathbf{f}_s$ for all $s \in I$, where I is a countable index set. If $A = URU^*$, then TFAE:*

- (i) $\cup_{s \in I} \{A^j \mathbf{f}_s\}_{j=0}^{L_s}$ is a scalable frame for \mathbb{H} ,
- (ii) $\cup_{s \in I} \{R^j \mathbf{v}_s\}_{j=0}^{L_s}$ is a scalable frame for \mathbb{H} .

Corollary 3 is relevant to the Schur decomposition: recall, any operator A on a finite-dimensional Hilbert spaces \mathbb{H} has a non-unique Schur decomposition of type $A = URU^*$, where U is a unitary $n \times n$ matrix, and R is of Schur form. When $A = A^*$, i.e., A is normal, then the Schur decomposition becomes unique, and is reduced to the classical unitary diagonalization. In the next subsection, we exploit the simplicity of the unitary diagonalization of normal operators to give more explicit conditions on the normal operator A in order to ensure scalability of a frame of type $F_G^{\mathbf{L}}(A)$.

4.1. Normal operators. Let A be a normal operator on \mathbb{H} . By the spectral theorem, there exists a unitary operator U , and a diagonal operator D such that $A = UDU^*$; in fact, for each $j \in \mathbb{Z}_+$ $A^j = UD^jU^*$.

Now, let $\mathcal{G} = \{\mathbf{f}_s\}_{s \in I}$ and set $\mathbf{v}_s = U^*\mathbf{f}_s$, $s \in I$. Then for each $j \in \mathbb{Z}_+$,

$$A^j\mathbf{f}_s = UDU^jU^*\mathbf{f}_s = UD^j\mathbf{v}_s = U(D^j\mathbf{v}_s) \quad \text{for all } \mathbf{f}_s \in \mathcal{G}. \quad (16)$$

Corollary 3 for normal operators reads as follows:

Corollary 4. *Let A be a normal operator on \mathbb{H} , and let $A = UDU^*$ be its unitary diagonalization. Let $\{\mathbf{f}_s\}_{s \in I} \subset \mathbb{H}$, and set $\mathbf{v}_s = U^*\mathbf{f}_s$, $s \in I$. TFAE*

- (i) *The set $\cup_{s \in I} \{A^j\mathbf{f}_s \mid j = 0, 1, \dots, L_s\}$ is a scalable frame for \mathbb{H} .*
- (ii) *The set $\cup_{s \in I} \{D^j\mathbf{v}_s \mid j = 0, 1, \dots, L_s\}$ is a scalable frame for \mathbb{H} .*

We now restrict our attention to a finite dimensional Hilbert space $\mathbb{H} = \mathbb{R}^n$ or \mathbb{C}^n . Let us first point out that the frame scalability property is preserved under simple manipulations:

Proposition 1. *Let $F = \{\mathbf{f}_i\}_{i=1}^k$ be a scalable frame for \mathbb{H} , $\dim H = n$. Then the following are also scalable frames:*

- (i) *any column or row permutation of F*
- (ii) *$\{U\mathbf{f}_i\}_{i=1}^k$ for any unitary matrix U*

Given a diagonal operator D in a Hilbert space \mathbb{H} with $\dim \mathbb{H} = n$, we first focus our attention on solving the *one-vector problem*: we look for conditions on D , and an unknown vector $\mathbf{v} \in \mathbb{H}$, which generate a scalable frame for \mathbb{H} of type (1).

Let $L \geq 0$, let D denote a diagonal $n \times n$ matrix, with diagonal entries a_1, \dots, a_n , and let $\mathbf{v} = (x(1), \dots, x(n))^T \in \mathbb{H}$. Let $w_j \in \mathbb{R}_+$, $0 \leq j \leq L$, be scaling coefficients such that $F_W = \{w_j D^j \mathbf{v}\}_{j=0}^L$ is a Parseval frame for \mathbb{H} , i.e.,

$$F_W F_W^* = I. \quad (17)$$

Note that (17) is equivalent to the system of equations

$$\begin{aligned} |x(i)|^2 \sum_{k=0}^L w_k^2 |a_i|^{2k} &= 1, \quad i = 1, \dots, n; \\ \sum_{k=0}^L w_k^2 (a_i \bar{a}_j)^k &= 0, \quad i \neq j. \end{aligned} \quad (18)$$

There exist real solutions of (18) when $n \leq 2$. For instance, when $\mathbb{H} = \mathbb{R}^2$, the choice of $\mathbf{v} = (0.5, 0.5)^T$ and $D = \text{diag}(1, -1)$ generates the set $\{\mathbf{v}, D\mathbf{v}, D^2\mathbf{v}, D^3\mathbf{v}\}$, which is a Parseval frame for \mathbb{R}^2 . However, when $\mathbb{H} = \mathbb{R}^3$, the equation $\sum_{k=0}^L w_k^2 (a_i \bar{a}_j)^k = 0$, $i \neq j$ implies that for the first three a_i 's, we always have the relation $a_1 a_2$, $a_1 a_3$, and $a_2 a_3$ are all negative numbers assuming $w_i \neq 0$, $i = 1, 2, 3$, which is not possible. Thus we have:

Theorem 3. *Let $\mathbf{v} \in \mathbb{R}^n$, and $a_1, \dots, a_n \in \mathbb{R}$. If $n \geq 2$, then any normal operator for \mathbb{R}^n can not generate a strictly scalable frame from \mathbf{v} .*

In contrast to the real case, there exists a solution to the one-vector problem in \mathbb{C}^n , involving the k -th root of unity:

Example 1. Let $\gamma = e^{2\pi i/k}$, $k \geq n$. Then the following dynamical operator A and the vector \mathbf{v}

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \gamma^{n-1} \end{pmatrix}, \quad \mathbf{v} = \frac{1}{\sqrt{k}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

generate the Harmonic tight frame $F_{\mathbf{v}}^{k-1}$.

Next, we consider the multi-generator case: By (9), the scaling coefficients $w_{s,j}$ related to vectors $D^j \mathbf{v}_s$, $0 \leq j \leq L(s)$, where $\mathbf{v}_s = (x_s(1), \dots, x_s(n))^T$, $1 \leq s \leq p$, need to be solutions to the following system of equations:

$$\begin{cases} \sum_{s=1}^p |x_s(i)|^2 \left[w_{s,0}^2 + w_{s,1}^2 |a_i|^2 + \dots + w_{s,L_s}^2 |a_i|^{2L_s} \right] = 1, \\ \sum_{s=1}^p x_s(i) \bar{x}_s(j) \left[w_{s,0}^2 + w_{s,1}^2 a_i \bar{a}_j + \dots + w_{s,L_s}^2 (a_i \bar{a}_j)^{L_s} \right] = 0, \end{cases} \quad (19)$$

for all $i, j = 1, \dots, n$, $i \neq j$.

Proposition 2. Let D be a diagonal $n \times n$ matrix with diagonal entries $a_1, \dots, a_n \in \mathbb{C}$, and let $\mathbf{v}_s = (x_s(1), \dots, x_s(n))^T \in \mathbb{C}^n$, $s \in \{1, \dots, p\}$, $p \geq 1$. TFAE:

- (i) The set $\cup_{s=1}^p \{D^j \mathbf{v}_s \mid j = 0, 1, \dots, L_s\}$ is a scalable frame for \mathbb{H}
- (ii) There exist scaling coefficients $w_{s,0}, w_{s,1}, \dots, w_{s,L_s}$, $1 \leq s \leq p$, which satisfy conditions (19).

By Corollary 4 and Proposition 2, the following result holds true for a finite dimensional Hilbert space \mathbb{H} :

Theorem 4. Let $A = UDU^*$ be a normal $n \times n$ matrix, where U is unitary, and D is diagonal, with diagonal entries $a_1, \dots, a_n \in \mathbb{C}$. Let $\mathbf{f}_s \in \mathbb{H}$, and set $\mathbf{v}_s = U^* \mathbf{f}_s = (x_s(1), \dots, x_s(n))^T$, $1 \leq s \leq p$.

The set $\cup_{s=1}^p \{A^j \mathbf{f}_s \mid 0 \leq j \leq L_s\}$ is a scalable frame of \mathbb{H} if and only if there exists a positive solution $w_{s,0}, w_{s,1}, \dots, w_{s,L_s}$, $1 \leq s \leq p$ to the system of equations (19), defined with respect to a_1, \dots, a_n and $x_s(1), \dots, x_s(n)$, $1 \leq s \leq p$.

Comment 2. The problem of finding specific conditions under which the set in item (ii) in Corollary 3 is a scalable frame for \mathbb{H} is still open for operators which do not possess a unitary diagonalization. For this reason, we further study several operators with special structures, such as block-diagonal operators (section 5) and companion operators (subsection 6).

5. BLOCK-DIAGONAL OPERATORS

In this section, we explore the case when the operator A is of block-diagonal form. Block-diagonal operators give us a chance to offer a partial answer to (Q1) in the case when we don't have a unitary diagonalization.

Note that in subsection 5.1 we give examples of operators which generate scalable frames in Hilbert spaces of dimension 2 and 3. Since we can treat \mathbb{H} with $\dim \mathbb{H} = n$ as a decomposition of several subspaces of dimensions 2 and 3, the examples in subsection 5.1 provide infinite examples of block-diagonal operators which generate scalable frames for \mathbb{H} .

Theorem 5. *Let F_s be a scalable frame for \mathbb{H}_s , with $\dim \mathbb{H}_s = n_s$, $s = 1, \dots, p$, and let*

$$G = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & F_p \end{pmatrix}. \quad (20)$$

Then G is a scalable frame for $\mathbb{H} = \mathbb{H}_1 \oplus \dots \oplus \mathbb{H}_p$.

Definition 1. *Let $A_s : \mathbb{H}_s \rightarrow \mathbb{H}_s$ be an operator on \mathbb{H}_s , with $\dim \mathbb{H}_s = n_s$, $1 \leq s \leq p$. Let $A : \mathbb{H}_s \rightarrow \mathbb{H}_s$ be a block-diagonal operator on $\mathbb{H} = \bigoplus_{s=1}^p \mathbb{H}_s$, constructed as follows:*

$$A = \begin{pmatrix} A_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & A_p \end{pmatrix}. \quad (21)$$

Let $\mathbf{v} \in \mathbb{H}_s$ for some $1 \leq s \leq p$. We say that \mathbf{v} is well-embedded in $\mathbf{f} \in \mathbb{H}$ with respect to operator (21) if

$$\begin{cases} \mathbf{f}(j) = \mathbf{v}(i), & \text{if } j = n_1 + \dots + n_s + i \\ \mathbf{f}(j) = 0, & \text{otherwise.} \end{cases} \quad (22)$$

Whenever \mathbf{v} is well-embedded in \mathbf{f} with respect to (21), we have

$$A\mathbf{f} = \begin{pmatrix} 0 \\ A_s \mathbf{v} \\ 0 \end{pmatrix}.$$

Theorem 6. *Let $A_s : \mathbb{H}_s \rightarrow \mathbb{H}_s$ be an operator on \mathbb{H}_s , with $\dim \mathbb{H}_s = n_s$, $1 \leq s \leq p$. Let $A : \mathbb{H}_s \rightarrow \mathbb{H}_s$ be a block-diagonal operator on $\mathbb{H} = \bigoplus_{s=1}^p \mathbb{H}_s$, constructed as in (21). Let $\mathbf{f}_{s,1}, \dots, \mathbf{f}_{s,m_s} \in \mathbb{H}$, $1 \leq s \leq p$ be well-embedded vectors $\mathbf{v}_{s,1}, \dots, \mathbf{v}_{s,m_s} \in \mathbb{H}_s$, $1 \leq s \leq p$.*

$$\text{The set } \bigcup_{s=1}^p \{A^j \mathbf{f}_{s,k} \mid 1 \leq k \leq m_s\}_{j=0}^{L_{s,k}} \quad (23)$$

is a (scalable) frame of \mathbb{H} if and only if $\{A_s^j \mathbf{v}_{s,k} \mid 1 \leq k \leq m_s\}_{j=0}^{L_{s,k}}$ are (scalable) frames of \mathbb{H}_s for all $1 \leq s \leq p$.

Proof. We assume that all $m_s = 1$, i.e., $\mathbf{f}_{s,k} = \mathbf{f}_s$, $\mathbf{v}_{s,k} = \mathbf{v}_s$, and $L_{s,k} = L_s$, $1 \leq s \leq p$, to simplify the presentation of the proof. The matrix

representation of $\cup_{s=1}^p \{A^j \mathbf{f}_s\}_{j=0}^{L_s}$ with scaling coefficients $w_{s,j}$, $0 \leq j \leq L_s$ for each $s = 1, \dots, p$ is of block-diagonal form:

$$F = \begin{pmatrix} w_{1,0} \mathbf{v}_1 & \dots & w_{1,L_1} A_1^{L_1} \mathbf{v}_1 & & \\ & & & \ddots & \\ & & & & w_{p,0} \mathbf{v}_p & \dots & w_{p,L_p} A_p^{L_p} \mathbf{v}_p \end{pmatrix}.$$

If F is a tight frame, then row vectors of F are orthogonal and have the same norm and so does $(w_{s,0} \mathbf{v}_s \dots w_{s,L_s} A_s^{L_s} \mathbf{v}_s)$ for each $s = 1, \dots, p$. This implies that the system $\{A_s^j \mathbf{v}_s\}_{j=0}^{L_s}$ is a scalable frame for \mathbb{H}_s for all $1 \leq s \leq p$.

Now, suppose that for each $1 \leq s \leq p$, the system $\{A_s^j \mathbf{v}_s\}_{j=0}^{L_s}$ is a scalable frame for \mathbb{H}_s . Then, there exist some scaling coefficients $w_{s,j}$, $1 \leq s \leq p$, $0 \leq j \leq L_s$, such that $\{w_{s,j} A_s^j \mathbf{v}_s | 0 \leq j \leq L_s\}$ is a Parseval frame for each $s = 1, \dots, p$. \square

5.1. Scalable dynamical frames for \mathbb{R}^2 and \mathbb{R}^3 . For the classification of a tight frame in this section, we use the notion of the *diagram vector*. For any $\mathbf{f} \in \mathbb{R}^n$, we define the diagram vector associated with \mathbf{f} , denoted $\tilde{\mathbf{f}}$, by

$$\tilde{\mathbf{f}} = \frac{1}{\sqrt{n-1}} \begin{pmatrix} \mathbf{f}(1)^2 - \mathbf{f}(2)^2 \\ \vdots \\ \mathbf{f}(n-1)^2 - \mathbf{f}(n)^2 \\ \sqrt{2n} \mathbf{f}(1) \mathbf{f}(2) \\ \vdots \\ \sqrt{2n} \mathbf{f}(n-1) \mathbf{f}(n) \end{pmatrix} \in \mathbb{R}^{n(n-1) \times 1},$$

where the difference of squares $\mathbf{f}(i)^2 - \mathbf{f}(j)^2$ and the product $\mathbf{f}(i) \mathbf{f}(j)$ occur exactly once for $i < j$, $i = 1, 2, \dots, n-1$.

Analogously, for any vector $\mathbf{f} \in \mathbb{C}^n$, we define the diagram vector associated with \mathbf{f} , denoted $\tilde{\mathbf{f}}$, by

$$\tilde{\mathbf{f}} = \frac{1}{\sqrt{n-1}} \begin{pmatrix} \mathbf{f}(1) \overline{\mathbf{f}(1)} - \mathbf{f}(2) \overline{\mathbf{f}(2)} \\ \vdots \\ \mathbf{f}(n-1) \overline{\mathbf{f}(n-1)} - \mathbf{f}(n) \overline{\mathbf{f}(n)} \\ \sqrt{n} \mathbf{f}(1) \overline{\mathbf{f}(2)} \\ \sqrt{n} \mathbf{f}(1) \mathbf{f}(2) \\ \vdots \\ \sqrt{n} \mathbf{f}(n-1) \overline{\mathbf{f}(n)} \\ \sqrt{n} \mathbf{f}(n-1) \mathbf{f}(n) \end{pmatrix} \in \mathbb{C}^{3n(n-1)/2},$$

where the difference of the form $\mathbf{f}(i) \overline{\mathbf{f}(i)} - \mathbf{f}(j) \overline{\mathbf{f}(j)}$ occurs exactly once for $i < j$, $i = 1, 2, \dots, n-1$ and the product of the form $\mathbf{f}(i) \overline{\mathbf{f}(j)}$ occurs exactly once for $i \neq j$.

The diagram vectors give us the following characterizations of tight frames and scalable frames:

Theorem 7. [17, 16] Let $\{\mathbf{f}_i\}_{i=1}^k$ be a sequence of vectors in \mathbb{H} , not all of which are zero. Then $\{\mathbf{f}_i\}_{i=1}^k$ is a tight frame if and only if $\sum_{i=1}^k \tilde{\mathbf{f}}_i = 0$.

Theorem 8. [17, 16] Let $\{\mathbf{f}_i\}_{i=1}^k$ be a unit-norm frame for \mathbb{H} and c_1, \dots, c_k be nonnegative numbers, which are not all zero. Let \tilde{G} be the Gramian associated to the diagram vectors $\{\tilde{\mathbf{f}}_i\}_{i=1}^k$. Then $\{c_i \mathbf{f}_i\}_{i=1}^k$ is a tight frame for \mathbb{H} if and only if $\mathbf{f} = (c_1^2 \dots c_k^2)^T$ belongs to the null space of \tilde{G} .

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard orthonormal basis in \mathbb{R}^n or \mathbb{C}^n .

Proposition 3. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be an operator in \mathbb{R}^2 , where a, b, c, d are not all zeros. If $a = 0$ and $b \neq 0$, then $F_{\mathbf{e}_1}^1$ is a scalable frame for \mathbb{R}^2 .

Proof. If $a = 0$ and $b \neq 0$, then $F_{\mathbf{e}_1}^1 = \{(1, 0)^T, (0, b)^T\}$. Since the two vectors in $F_{\mathbf{e}_1}^1$ are orthogonal, $F_{\mathbf{e}_1}^1$ is a strictly scalable frame for \mathbb{R}^2 . \square

We highlight that, when $b = d \neq 0$ and $c = -d/4$ in Proposition 3, the matrix A is non-diagonalizable yet generates a scalable frame for \mathbb{R}^2 .

Proposition 4. Let a, b, c, d be real numbers such that $a \neq -d$,

$$b = \frac{\pm 1}{a+d} \sqrt{\frac{a^2(a+d)^2 + (a+d)^2 + a^2}{1 + (a+d)^2}}, \text{ and}$$

$$c = \mp a(ad + a^2 + 1) \sqrt{\frac{1 + (a+d)^2}{(a+d)^2 + a^2(a+d)^2 + a^2}}.$$

Then the operator $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ in \mathbb{R}^2 generates a tight frame

$$F_{\mathbf{e}_1}^2 = \begin{pmatrix} 1 & a & a^2 + bc \\ 0 & b & ab + bd \end{pmatrix}.$$

Theorem 9. Let a, b, c, d be real numbers such that $a > 0$ and $abcd \neq 0$. Then the following two statements are equivalent:

- (1) $0 < -\frac{ac}{bd} < 1$.
- (2) The system

$$F = \begin{pmatrix} 1 & a & c \\ 0 & b & d \end{pmatrix}$$

is a strictly scalable frame for \mathbb{R}^2 .

Proof. We first note that the condition $0 < -\frac{ac}{bd} < 1$ is equivalent to $(a > 0, -\frac{b}{c} > \frac{a}{d} > 0)$ or $(a > 0, -\frac{d}{a} > \frac{c}{b} > 0)$.

(1) \Rightarrow (2): The conditions $a > 0, -\frac{b}{c} > \frac{a}{d} > 0$ imply that

$$d > 0, ad - bc > 0, \frac{ac}{bd} > -1$$

and the conditions $a > 0$, $-\frac{d}{a} > \frac{c}{b} > 0$ imply that

$$d < 0, ad - bc < 0, \frac{ac}{bd} > -1.$$

Then

$$x = \sqrt{\frac{ac}{bd} + 1}, y = \sqrt{\frac{c}{-b(ad - bc)}}, z = \sqrt{\frac{a}{d(ad - bc)}}$$

are positive numbers and

$$F = \begin{pmatrix} x & ya & zc \\ 0 & yb & zd \end{pmatrix}$$

is a Parseval frame for \mathbb{R}^2 .

(1) \Leftarrow (2): It the system F is strictly scalable, then the normalized system

$$F' = \begin{pmatrix} 1 & \frac{a}{\sqrt{a^2+b^2}} & \frac{c}{\sqrt{c^2+d^2}} \\ 0 & \frac{b}{\sqrt{a^2+b^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix}$$

is a unit-norm scalable frame. By Theorem 8, the Gramian matrix of diagram vectors of F' has positive scalings in its null space:

$$\frac{a^2cd - abc^2 + abd^2 - b^2cd}{ab(c^2 + d^2)} > 0, \quad (24)$$

$$\frac{-cd(a^2 + b^2)}{ab(c^2 + d^2)} > 0. \quad (25)$$

Inequality (25) implies that $-\frac{ac}{bd} > 0$. Next we show that $-\frac{ac}{bd} < 1$.

In case $b > 0$, inequality (24) implies that

$$a^2cd + abd^2 > bc(ac + bd).$$

If ($c > 0$ and $ac + bd \geq 0$) or ($c < 0$ and $ac + bd \leq 0$), then $a^2cd + abd^2 > 0$, which implies $-\frac{ac}{bd} < 1$. If $c > 0$ and $ac + bd < 0$, then $ac < -bd$, which implies $1 < -\frac{bd}{ac}$ since $ac > 0$. Similarly, if $c < 0$ and $ac + bd > 0$, then $ac > -bd$, which implies $1 < -\frac{bd}{ac}$ since $ac < 0$. This is equivalent to $-\frac{ac}{bd} < 1$.

In case $b < 0$, suppose that $-\frac{ac}{bd} \geq 1$. Multiply both sides by the positive number $-abd^2$. On one hand we have $a^2cd \geq -abd^2$ and on the other hand, from inequality (24), we have $a^2cd - abc^2 < -abd^2 + b^2cd$. Since $a^2cd \geq -abd^2$, we have $-abd^2 - abc^2 < -abd^2 + b^2cd$, which implies $-\frac{ac}{bd} < 1$. This contradicts our assumption. \square

This observation provides us the conditions for a dynamical operator A in \mathbb{R}^2 to generate a scalable frame $F_{\mathbf{e}_1}^2$ for \mathbb{R}^2 .

Corollary 5. *Let a, b, c, d be real numbers such that $a > 0$ and $0 < -\frac{a(a^2+bc)}{b^2(a+d)} < 1$.*

1. *Then the operator $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ generates a strictly scalable frame*

$$F_{\mathbf{e}_1}^2 = \begin{pmatrix} 1 & a & a^2 + bc \\ 0 & b & ab + bd \end{pmatrix}.$$

If $2\sin^2(\omega) - 1 > 0$, then the operator

$$A = \begin{pmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix}$$

satisfies the condition on Theorem 9. Consequently we have:

Example 2. *Let*

$$A = \begin{pmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix},$$

where $2\sin^2(\omega) - 1 > 0$. Then the operator A generates a strictly scalable frame

$$F_{\mathbf{e}_1}^2 = \begin{pmatrix} 1 & \cos(\omega) & \cos(2\omega) \\ 0 & \sin(\omega) & \sin(2\omega) \end{pmatrix}.$$

Proposition 5. *Let a, b, c, d be real numbers such that $abcd < 0$. Then the system*

$$F = \begin{pmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{pmatrix}$$

is a strictly scalable frame for \mathbb{R}^2 .

Proof. We define

$$p = \sqrt{\left(\frac{acd}{b} - c^2\right)s^2 + 1}, q = \sqrt{\left(\frac{bcd}{a} + d^2\right)s^2 + 1}, r = \sqrt{-\frac{cd}{ab}}.$$

For any a, b, c, d such that $abcd < 0$, one can select s such that $p > 0$ and $q > 0$. Those choices of p, q, r, s guarantee that the system

$$F = \begin{pmatrix} p & 0 & ra & sc \\ 0 & q & rb & sd \end{pmatrix}$$

is a Parseval frame. □

Corollary 6. *Let a, b be real numbers such that $a + b^2 < 0$. Then the operator $A = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$ generates a strictly scalable frame $F_{\mathbf{e}_1}^3$ for \mathbb{R}^2 .*

We next explore when a dynamical operator A generates a scalable frame $F_{\mathbf{e}_1}^3$ in \mathbb{R}^3 . We first observe the following systems in \mathbb{R}^3 when $ab \neq 0$

$$F1 = \begin{pmatrix} 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & a & c \\ 0 & 0 & 1 & b & d \end{pmatrix}, \quad F2 = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & a & c \\ 0 & 0 & b & d \end{pmatrix}. \quad (26)$$

If F is a tight frame, by Theorem 7, we have

$$\begin{aligned} ax + cy &= 0 \\ bx + dy &= 0 \\ ab + cd &= 0, \end{aligned} \quad (27)$$

which implies that $x = y = 0$. That is, the last two vectors have only two nonzero elements in the same entries.

We note that if the first column of A is \mathbf{e}_1 , then the system $F_{\mathbf{e}_1}^3$ can not be a frame for \mathbb{R}^3 . Let

$$A = \begin{pmatrix} 0 & a & x \\ 1 & b & y \\ 0 & c & z \end{pmatrix}. \quad (28)$$

Then the corresponding F_4 system has the following entries:

$$F_{\mathbf{e}_1}^3 = \begin{pmatrix} 1 & 0 & a & ab + cx \\ 0 & 1 & b & b^2 + cy + a \\ 0 & 0 & c & bc + cz \end{pmatrix}.$$

By (27), for the system $F_{\mathbf{e}_1}^3$ to be a strictly scalable frame, we need to assume $a = ab + cx = 0$ or $b = b^2 + cy + a = 0$. We first consider the case $a = ab + cx = 0$.

Proposition 6. *Let a, b, c, d be real numbers such that $a > 0$ and $0 < -\frac{a(a^2+bc)}{b^2(a+d)} < 1$. Then the operator*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & a & c \\ 0 & b & d \end{pmatrix} \quad (29)$$

generates a strictly scalable frame

$$F_{\mathbf{e}_1}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & a^2 + bc \\ 0 & 0 & b & ab + bd \end{pmatrix}. \quad (30)$$

Proof. This follows from Theorem 5 and Theorem 9. \square

When $b = b^2 + cy + a = 0$, we have

$$A = \begin{pmatrix} 0 & a & x \\ 1 & 0 & -a/c \\ 0 & c & cz \end{pmatrix}.$$

By applying row and column permutations, $F_{\mathbf{e}_1}^3$ can be written in the same form as (30). Similarly, the following operator, with a suitable choice of the second and third column:

$$A = \begin{pmatrix} 0 & a & x \\ 0 & b & y \\ 1 & c & z \end{pmatrix} \quad (31)$$

generates a scalable frame $F_{\mathbf{e}_1}^3$, which also can be written in the same form as (30).

We note that any tight or scalable frame in \mathbb{R}^n with n frame vectors is an orthogonal basis. A trivial example of a scalable dynamical frame is the following:

Example 3. *Let*

$$A = \begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \end{pmatrix}. \quad (32)$$

Then the sequence $F_{\mathbf{e}_1}^L$ is a scalable frame of \mathbb{R}^n if and only if $L \geq n$.

For instance, when $n = L = 3$, the resulting frame is $F_{\mathbf{e}_1}^3 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\}$, and the scaled frame $\{2^{-1/2}\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, 2^{-1/2}\mathbf{e}_1\}$ is a Parseval frame.

Notice that (32) is an example of a companion [22] operator. It makes sense to explore the conditions under which a companion operator generates a scalable frame.

6. COMPANION OPERATORS AND GENERALIZATIONS

Let $a_1, \dots, a_n \in \mathbb{R}$ which are not all zeros, then

$$A = \left(\begin{array}{c|c} 0 & a_1 \\ \hline & a_2 \\ I_{n-1} & \vdots \\ & a_n \end{array} \right) \quad (33)$$

is called a companion operator [22].

Proposition 7. *Let the dynamical operator A be a companion operator (33) in \mathbb{R}^n , then we have*

- (1) $F_{\mathbf{e}_1}^{n-1} = I$.
- (2) *for any orthogonal matrix U , the operator UAU^{-1} generates an orthonormal basis U .*

It is known that the standard orthonormal basis B can not be extended to a scalable frame by adding one vector $\mathbf{f} \in \mathbb{H} \setminus B$, [20, 24]. Thus we explore when one can generate a dynamical frame by adding two vectors. Although a companion operator A does not generate a scalable frame $F_{\mathbf{e}_1}^n$, it can generate a scalable frame $F_{\mathbf{e}_1}^{n+1}$ under certain conditions. Using the companion operator A , we have

$$F_{\mathbf{e}_1}^n = (\mathbf{e}_1 \dots \mathbf{e}_n \mathbf{f}), \quad F_{\mathbf{e}_1}^{n+1} = (\mathbf{e}_1 \dots \mathbf{e}_n \mathbf{f} \mathbf{g}), \quad (34)$$

where

$$\mathbf{f} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} a_1 a_n \\ a_1 + a_2 a_n \\ a_2 + a_3 a_n \\ \vdots \\ a_{n-2} + a_{n-1} a_n \\ a_{n-1} + a_n^2 \end{pmatrix}.$$

Similar calculations as in observation (27) produce the following result:

Proposition 8. [20] *Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard orthonormal basis in \mathbb{R}^n with $n \geq 2$. Let \mathbf{f} and \mathbf{g} be two unit-norm vectors in \mathbb{R}^n .*

If either system $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}, \mathbf{g}\}$ or $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \mathbf{f}, \mathbf{g}\}$ is scalable, then \mathbf{f} and \mathbf{g} have only two nonzero elements in the same entries.

We now assume that $F_{\mathbf{e}_1}^{n+1}$ is scalable. Then by Proposition 8, $a_m = 0$ implies that $a_{m-1} = 0$ for $m \geq 2$. This implies that $a_1 = \dots = a_{n-2} = 0$.

Proposition 9. *Let a and b be real numbers such that $a > 0$ and $0 < -\frac{a^2}{a+b^2} < 1$. Then the companion operator A in \mathbb{R}^n ,*

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & \dots & 1 & 0 & a \\ 0 & 0 & \dots & 0 & 1 & b \end{pmatrix} \quad (35)$$

generates a strictly scalable frame $F_{\mathbf{e}_1}^{n+1}$.

Proof. We have

$$F_{\mathbf{e}_1}^{n+1} = \begin{pmatrix} I_{n-2} & & & & \\ & 1 & 0 & a & ab \\ & 0 & 1 & b & a+b^2 \end{pmatrix}. \quad (36)$$

The strict scalability follows from Theorem 9 and Theorem 5. \square

We note that the operator A in (35) is not diagonalizable. Next, we generalize the structure of A while ensuring that the new matrix generates scalable frames by iterative actions.

Example 4. *Let a and b be real numbers such that $0 < -\frac{a(a^2+bc)}{b^2(a+d)} < 1$ and $a > 0$. Then the operator*

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & a & c \\ 0 & \dots & 0 & b & d \end{pmatrix} \quad (37)$$

generates a strictly scalable frame $F_{\mathbf{e}_1}^n$ for \mathbb{R}^n .

Proof. We have

$$F_{\mathbf{e}_1}^n = \begin{pmatrix} I_{n-2} & & & & \\ & 1 & a & a^2+bc \\ & 0 & b & ab+bd \end{pmatrix}. \quad (38)$$

The strict scalability follows by Proposition 4 and Proposition 5. \square

Example 5. Let $2 \sin^2(\omega) - 1 > 0$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \cos(\omega) & -\sin(\omega) \\ 0 & \dots & 0 & \sin(\omega) & \cos(\omega) \end{pmatrix} \quad (39)$$

generates a strictly scalable frame $F_{\mathbf{e}_1}^n$.

Example 6. Let $2 \sin^2(\phi) - 1 > 0$ and let

$$A = \begin{pmatrix} \pm 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \pm 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pm 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \dots & 0 & \sin \phi & \cos \phi \end{pmatrix}. \quad (40)$$

The set

$$\{\mathbf{e}_{n-1}, A\mathbf{e}_{n-1}, A^2\mathbf{e}_{n-1}\} \cup \bigcup_{l=1}^{n-2} \{\mathbf{e}_l, A\mathbf{e}_l, \dots, A^{L_l}\mathbf{e}_l\} \quad (41)$$

is a strictly scalable frame of \mathbb{R}^n .

7. CONCLUDING REMARKS AND GENERALIZATIONS

We have studied the scalability of dynamical frames in a separable Hilbert space \mathbb{H} . Given an operator A on \mathbb{H} and a (at most countable) set $G \subset \mathbb{H}$, we have explored the relations between A , G and the number of iterations that make the system (1) a scalable frame. When $\dim \mathbb{H}$ is finite, and A is a normal operator, we have fully answered question (Q1).

Since we have not achieved a full answer for operators which are not unitary diagonalizable, we have offered a partial answer by studying block-diagonal operators, which are not necessarily normal. Note that the block-diagonal matrix A in Theorem 6 cannot be normal if one of its blocks is not normal. Also, we have established the canonical dual frame for frames of type $F_G(A)$; in particular, we showed that the canonical dual frame has, as anticipated, an iterative set structure. This result holds true in any separable Hilbert space \mathbb{H} .

We now pose a new question, which is a generalization of (Q1):

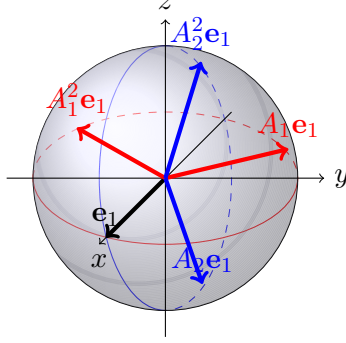
(Q3) Given multiple operators A_s , $s \in I$ on a separable Hilbert space \mathbb{H} , and one fixed vector $\mathbf{v} \in \mathbb{H}$, when is the system $\cup_{s \in I} \{A_s^j \mathbf{v}\}_{j=0}^{L_s}$ a (scalable) frame for \mathbb{H} ?

The next example shows how to generate a scalable frame for \mathbb{R}^3 using two dynamical operators.

Example 7. Let $\alpha = 2\pi/3$,

$$A_1 = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 0 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix}.$$

Then $\{\mathbf{e}_1, A_1\mathbf{e}_1, A_1^2\mathbf{e}_1, A_2\mathbf{e}_1, A_2^2\mathbf{e}_1\}$ is a strictly scalable frame for \mathbb{R}^3 .



The following proposition is a generalization of the principle introduced in Example 7:

Proposition 10. Let $i, j, k, l \in \mathbb{N}$ be such that $p < k \leq n$, $q < l \leq n$, and let $N \in \mathbb{N}$. For each $m = 1, \dots, N$, we define $A_{kl}^{pq}(m) = [a_{ij}(m)]_{i,j=1}^n$ as

$$a_{pq}(m) = a_m, a_{pl}(m) = b_m, a_{kq}(m) = c_m, a_{kl}(m) = d_m.$$

If for each $m = 1, \dots, N$, a_m, b_m, c_m and d_m satisfy the conditions of Corollary 9, and the system

$$\{\mathbf{e}_1\} \cup \bigcup_{m=1}^N \{A_{kl}^{pq}(m)\mathbf{e}_1, (A_{kl}^{pq}(m))^2\mathbf{e}_1\} \quad (42)$$

spans \mathbb{R}^n , then (42) is a strictly scalable frame for \mathbb{R}^n .

Proof. By Corollary 9, the set $\{\mathbf{e}_1\} \cup \{A_{kl}^{pq}(m)\mathbf{e}_1, (A_{kl}^{pq}(m))^2\mathbf{e}_1\}$ is a scalable frame for a 2-dimensional subspace for each $m = 1, \dots, N$. Thus, there exist some suitable scaling coefficients $x(m), y(m), z(m)$, and by Theorem 7,

$$\widetilde{x(m)\mathbf{e}_1} + y(m)\widetilde{A_{kl}^{pq}(m)\mathbf{e}_1} + z(m)\widetilde{(A_{kl}^{pq}(m))^2\mathbf{e}_1} = 0.$$

This implies that the system (42) is a scalable frame for \mathbb{R}^n . \square

For a frame generated by iterative actions of multiple operators, that is, a *multi-dynamical* frame, we find that its canonical dual frame is also multi-dynamical:

Theorem 10. Let A_s , $s \in I$, be operators on a separable Hilbert space \mathbb{H} , let $L_s \geq 0$, and fix a vector $\mathbf{v} \in \mathbb{H}$. If $\bigcup_{s \in I} \{A_s^j \mathbf{v}\}_{j=0}^{L_s}$ is a frame for \mathbb{H} , with frame operator S , then its canonical dual frame is

$$\bigcup_{s \in I} \{B_s^j \mathbf{f}\}_{j=0}^{L_s}, \quad (43)$$

where $\mathbf{f} = S^{-1}\mathbf{v}$, and $B_s = S^{-1}A_sS$, $s \in I$.

Proof. If S denotes the frame operator of the frame $\cup_s \{A_s^j \mathbf{v}\}_{j=0}^{L_s}$ for \mathbb{H} , then its canonical dual frame elements are $S^{-1}A_s^j \mathbf{v}$. Since $B_s^j = S^{-1}A_s^j S$, we obtain that the dual frame elements are

$$S^{-1}A_s^j \mathbf{v} = S^{-1}A_s^j S S^{-1} \mathbf{v} = S^{-1}A_s^j S \mathbf{f} = B_s^j \mathbf{f}.$$

□

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